

The Shi arrangement of the type D_ℓ

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Abstract: In this paper, we give a basis for the derivation module of the cone over the Shi arrangement of the type D_ℓ explicitly.

Key words: Hyperplane arrangement; Shi arrangement; Free arrangement.

1. Introduction. Let V be an ℓ -dimensional vector space. An *affine arrangement of hyperplanes* \mathcal{A} is a finite collection of affine hyperplanes in V . If every hyperplane $H \in \mathcal{A}$ goes through the origin, then \mathcal{A} is called to be *central*. When \mathcal{A} is central, for each $H \in \mathcal{A}$, choose $\alpha_H \in V^*$ with $\ker(\alpha_H) = H$. Let S be the algebra of polynomial functions on V and let Der_S be the module of derivations

$$\text{Der}_S := \{\theta : S \rightarrow S \mid \theta(fg) = f\theta(g) + g\theta(f), f, g \in S, \theta \text{ is } \mathbb{R}\text{-linear}\}.$$

For a central arrangement \mathcal{A} , recall

$$D(\mathcal{A}) := \{\theta \in \text{Der}_S \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

We say that \mathcal{A} is a *free arrangement* if $D(\mathcal{A})$ is a free S -module. The freeness was defined in [15]. The Factorization Theorem[16] states that, for any free arrangement \mathcal{A} , the characteristic polynomial of \mathcal{A} factors completely over the integers.

Let $E = \mathbb{R}^\ell$ be an ℓ -dimensional Euclidean space with a coordinate system x_1, \dots, x_ℓ , and Φ be a crystallographic irreducible root system. Fix a positive root system $\Phi^+ \subset \Phi$. For each positive root $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, we define an affine hyperplane

$$H_{\alpha,k} := \{v \in V \mid (\alpha, v) = k\}.$$

In [10], J.-Y. Shi introduced the *Shi arrangement*

$$S(A_\ell) := \{H_{\alpha,k} \mid \alpha \in \Phi^+, 0 \leq k \leq 1\}$$

when the root system is of the type A_ℓ . This definition was later extended to the *generalized Shi arrangement* (e.g., [4])

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$$\mathcal{S}(\Phi) := \{H_{\alpha,k} \mid \alpha \in \Phi^+, 0 \leq k \leq 1\}.$$

Embed E into $V = \mathbb{R}^{\ell+1}$ by adding a new coordinate z such that E is defined by the equation $z = 1$ in V . Then, as in [7], we have the cone $\mathbf{cS}(\Phi)$ of $\mathcal{S}(\Phi)$

$$\mathbf{cS}(\Phi) := \{\mathbf{c}H_{\alpha,k} \mid \alpha \in \Phi^+, 0 \leq k \leq 1\} \cup \{z = 0\}.$$

In [18], M. Yoshinaga proved that the cone $\mathbf{cS}(\Phi)$ is a free arrangement with exponents $(1, h, \dots, h)$ (h appears ℓ times), where h is the Coxeter number of Φ . (He actually verified the conjecture by P. Edelman and V. Reiner in [4], which is far more general.) He proved the freeness without finding a basis.

In [13], for the first time, the authors gave an explicit construction of a basis for $D(\mathbf{cS}(A_\ell))$. Then D. Suyama constructed bases for $D(\mathbf{cS}(B_\ell))$ and $D(\mathbf{cS}(C_\ell))$ in [14]. In this paper, we will give an explicit construction of a basis for $D(\mathbf{cS}(\mathcal{D}_\ell))$. A defining polynomial of the cone over the Shi arrangement of the type D_ℓ is given by

$$Q := z \prod_{1 \leq s < t \leq \ell} \prod_{\epsilon \in \{-1, 1\}} (x_s + \epsilon x_t - z)(x_s + \epsilon x_t).$$

Note that the number of hyperplanes in $\mathbf{cS}(\mathcal{D}_\ell)$ is equal to $2\ell(\ell - 1) + 1$. Our construction is similar to the construction in the case of the type B_ℓ . The essential ingredients of the recipe are the Bernoulli polynomials and their relatives.

2. The basis construction.

Proposition 2.1. For $(p, q) \in \mathbb{Z}_{\geq -1} \times \mathbb{Z}_{\geq 0}$, consider the following two conditions for a rational function $B_{p,q}(x)$:

1. $B_{p,q}(x+1) - B_{p,q}(x) = \frac{(x+1)^p - (-x)^p}{(x+1) - (-x)}(x+1)^q(-x)^q,$
2. $B_{p,q}(-x) = -B_{p,q}(x).$

Then such a rational function $B_{p,q}(x)$ uniquely exists. Moreover, the $B_{p,q}(x)$ is a polynomial unless $(p, q) = (-1, 0)$ and $B_{-1,0}(x) = -(1/x)$.

Proof. Suppose $(p, q) \neq (-1, 0)$. Since the right hand side of the first condition is a polynomial in x , there exists a polynomial $B_{p,q}(x)$ satisfying the first condition. Note that $B_{p,q}(x)$ is unique up to a constant term. Define a polynomial $F(x) = B_{p,q}(x) + B_{p,q}(-x)$. Since

$$\begin{aligned} & B_{p,q}(-x) - B_{p,q}(-x-1) \\ &= \frac{(-x)^p - (x+1)^p}{(-x) - (x+1)} (-x)^q (x+1)^q \\ &= \frac{(x+1)^p - (-x)^p}{(x+1) - (-x)} (x+1)^q (-x)^q \\ &= B_{p,q}(x+1) - B_{p,q}(x), \end{aligned}$$

we have $F(x+1) = F(x)$ for any x . Therefore $F(x)$ is a constant function. Then the polynomial $B_{p,q}(x) - (F(0)/2)$ is the unique solution satisfying the both conditions. Next we suppose $(p, q) = (-1, 0)$. Then we compute

$$\begin{aligned} & B_{-1,0}(x+1) - B_{-1,0}(x) \\ &= \frac{(x+1)^{-1} - (-x)^{-1}}{(x+1) - (-x)} = -\frac{1}{x+1} + \frac{1}{x}. \end{aligned}$$

Thus $B_{-1,0}(x) = -(1/x)$ is the unique solution satisfying the both conditions. \square

Definition 2.2. Define a rational function $\overline{B}_{p,q}(x, z)$ in x and z by

$$\overline{B}_{p,q}(x, z) := z^{p+2q} B_{p,q}(x/z).$$

Then $\overline{B}_{p,q}(x, z)$ is a homogeneous polynomial of degree $p + 2q$ except the two cases: $\overline{B}_{-1,0}(x, z) = -(1/x)$ and $\overline{B}_{0,q}(x, z) = 0$.

For a set $I := \{y_1, \dots, y_m\}$ of variables, let

$$\sigma_n^I := \sigma_n(y_1, \dots, y_m), \quad \tau_{2n}^I := \sigma_n(y_1^2, \dots, y_m^2),$$

where σ_n stands for the elementary symmetric function of degree n .

Definition 2.3. Define derivations

$$\begin{aligned} \varphi_j &:= (x_j - x_{j+1} - z) \sum_{i=1}^{\ell} \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} (\prod K_1) (\prod K_2)^2 \\ & \quad (-z)^{|K_1|} \sum_{\substack{0 \leq n_1 \leq |J_1| \\ 0 \leq n_2 \leq |J_2|}} (-1)^{n_1+n_2} \sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z) \frac{\partial}{\partial x_i} \end{aligned}$$

for $j = 1, \dots, \ell - 1$ and

$$\varphi_{\ell} := \sum_{i=1}^{\ell} \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} (\prod K_1) (\prod K_2)^2 (-z)^{|K_1|}$$

$$(-x_{\ell}) \overline{B}_{-1,k_0}(x_i, z) \frac{\partial}{\partial x_i}$$

for $j = \ell$, where

$$J := \{x_1, \dots, x_{j-1}\}, \quad J_1 := \{x_j, x_{j+1}\},$$

$$J_2 := \{x_{j+2}, \dots, x_{\ell}\},$$

$$\prod K_p := \prod_{x_i \in K_p} x_i \quad (p = 1, 2),$$

$$k_0 := |J \setminus (K_1 \cup K_2)| \geq 0,$$

$$k := (|J_1| - n_1) + 2(|J_2| - n_2) - 1 \geq -1.$$

Note that $\varphi_j(z) = 0$ ($1 \leq j \leq \ell$). In the rest of the paper, we will give a proof of the following theorem:

Theorem 2.4. The derivations $\varphi_1, \dots, \varphi_{\ell}$, together with the Euler derivation

$$\theta_E := z \frac{\partial}{\partial z} + \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i},$$

form a basis for $D(\mathbf{cS}(D_{\ell}))$.

Note that $\theta_E(x_i) = x_i$ ($1 \leq i \leq \ell$) and $\theta_E(z) = z$.

Lemma 2.5. Let $1 \leq i \leq \ell$ and $1 \leq j \leq \ell$. Suppose $\varphi_j(x_i)$ is nonzero. Then $\varphi_j(x_i)$ is a homogeneous polynomial of degree $2(\ell - 1)$.

Proof. Define

$$F_{ij} := (x_j - x_{j+1} - z) (\prod K_1) (\prod K_2)^2 z^{|K_1|} \sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z) \quad (1 \leq j \leq \ell - 1),$$

$$F_{i\ell} := (\prod K_1) (\prod K_2)^2 z^{|K_1|} x_{\ell} \overline{B}_{-1,k_0}(x_i, z)$$

when K_1, K_2, n_1, n_2 are fixed. Then $\varphi_j(x_i)$ is a linear combination of the F_{ij} 's over \mathbb{R} .

Note that $\overline{B}_{k,k_0}(x_i, z)$ is a polynomial unless $(k, k_0) = (-1, 0)$.

Assume that $1 \leq j \leq \ell - 1$ and $(k, k_0) = (-1, 0)$. Then $J = K_1 \cup K_2$, $n_1 = |J_1|$, $n_2 = |J_2|$, and $\overline{B}_{-1,0}(x_i, z) = -1/x_i$. Therefore each F_{ij} is a polynomial. Thus $\varphi_j(x_i)$ is a nonzero polynomial and there exists a nonzero polynomial F_{ij} . Compute

$$\deg \varphi_j(x_i) = \deg F_{ij}$$

$$= 1 + |K_1| + 2|K_2| + |K_1| + n_1 + 2n_2$$

$$\begin{aligned}
& + \deg \overline{B}_{k,k_0}(x_i, z) \\
& = 1 + 2|K_1| + 2|K_2| + n_1 + 2n_2 + (2k_0 + k) \\
& = 1 + 2|K_1| + 2|K_2| + n_1 + 2n_2 \\
& \quad + 2(|J| - |K_1| - |K_2|) + |J_1| - n_1 \\
& \quad + 2(|J_2| - n_2) - 1 \\
& = 2(|J| + |J_1| + |J_2|) - |J_1| = 2\ell - 2.
\end{aligned}$$

Next consider $\varphi_\ell(x_i)$. If $k_0 = 0$, then $J = K_1 \cup K_2$. Therefore each $F_{i\ell}$ is a polynomial. Thus so is $\varphi_\ell(x_i)$. Compute

$$\begin{aligned}
& \deg \varphi_\ell(x_i) \\
& = |K_1| + 2|K_2| + |K_1| + 1 + \deg \overline{B}_{-1,k_0}(x_i, z) \\
& = 2(|K_1| + |K_2|) + 1 + (2k_0 - 1) \\
& = 2(|K_1| + |K_2| + k_0) = 2(\ell - 1).
\end{aligned}$$

□

Let $<$ denote the *pure lexicographic order* of monomials with respect to the total order

$$x_1 > x_2 > \cdots > x_\ell > z.$$

When $f \in S = \mathbb{C}[x_1, x_2, \dots, x_\ell, z]$ is a nonzero polynomial, let $\text{in}(f)$ denote the *initial monomial* (e.g., see [6]) of f with respect to the order $<$.

Proposition 2.6. *Suppose $\varphi_j(x_i)$ is nonzero. Then*

- (1) $\text{in}(\varphi_j(x_i)) \leq x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$,
- (2) $\text{in}(\varphi_j(x_i)) < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$ for $i < j$,
- (3) $\text{in}(\varphi_i(x_i)) = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$ for $1 \leq i \leq \ell$.

Proof. Recall F_{ij} ($1 \leq j \leq \ell - 1$) and $F_{i\ell}$ from the proof of Lemma 2.5 when K_1, K_2, n_1, n_2 are fixed. Let $\deg^{(x_i)} f$ denote the degree of f with respect to x_i when $f \neq 0$.

(1) Since, for every nonzero F_{ij} , we obtain

$$\deg^{(x_p)} F_{ij} \leq 2 \quad (1 \leq p < i), \quad \deg(F_{ij}) = 2\ell - 2.$$

Hence we may conclude

$$\text{in}(F_{ij}) \leq x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$$

and thus

$$\text{in}(\varphi_j(x_i)) \leq \max\{\text{in}(F_{ij})\} \leq x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

(2) Suppose $i < j < \ell$. Since $x_i > x_j > z$, one has

$$\begin{aligned}
& \text{in}(\sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z)) \\
& \leq x_i^{n_1+2n_2+2k_0+k} = x_i^{2\ell-2j+2k_0-1}
\end{aligned}$$

when $\overline{B}_{k,k_0}(x_i, z)$ is nonzero. The equality holds if and only if $n_1 = n_2 = 0$.

Suppose that F_{ij} is nonzero. For $1 \leq i < j \leq \ell - 1$, we have

$$\begin{aligned}
& \text{in}(F_{ij}) \\
& = \text{in}(x_j - x_{j+1} - z) \text{in}((\prod K_1)(\prod K_2)^2(-z)^{|K_1|}) \\
& \quad \text{in}(\sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z)) \\
& \leq x_j \text{in}((\prod K_1)(\prod K_2)^2(-z)^{|K_1|}) x_i^{2\ell-2j+2k_0-1} \\
& = x_j \text{in}((\prod K_1)(\prod K_2)^2(-z)^{|K_1|} x_i^{2k_0}) x_i^{2\ell-2j-1} \\
& \leq x_j (x_1^2 \cdots x_{i-1}^2 x_i^{2j-2i}) x_i^{2\ell-2j-1} \quad (*) \\
& = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i-1} x_j < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.
\end{aligned}$$

Thus

$$\text{in}(\varphi_j(x_i)) < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

For $1 \leq i < j = \ell$,

$$\begin{aligned}
& \text{in}(F_{i\ell}) \\
& = x_\ell \text{in}((\prod K_1)(\prod K_2)^2(-z)^{|K_1|}) \text{in}(\overline{B}_{-1,k_0}(x_i, z)) \\
& = x_\ell \text{in}((\prod K_1)(\prod K_2)^2(-z)^{|K_1|}) x_i^{2k_0-1} \\
& = x_\ell \text{in}((\prod K_1)(\prod K_2)^2(-z)^{|K_1|} x_i^{2k_0}) x_i^{-1} \\
& \leq x_\ell (x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}) x_i^{-1} \quad (**) \\
& = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i-1} x_\ell < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.
\end{aligned}$$

This proves (2).

Now we only need to prove (3). Let $i = j < \ell$ in (*). Then the equality

$$\text{in}(F_{ii}) = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$$

holds if and only if

$$K_1 = \emptyset, K_2 = J, n_1 = n_2 = k_0 = 0, k = 2\ell - 2i - 1$$

because the leading term of $\overline{B}_{2\ell-2i-1,0}(x_i, z)$ is equal to

$$\frac{x_i^{2\ell-2i-1}}{2\ell-2i-1}.$$

Next let $i = \ell$ in (**). Then the equality

$$\text{in}(F_{\ell\ell}) = x_1^2 \cdots x_{\ell-1}^2$$

holds if and only if

$$K_1 = \emptyset, K_2 = J = \{x_1, \dots, x_{\ell-1}\}, k_0 = 0.$$

Therefore, for $1 \leq i \leq \ell$,

$$\text{in}(\varphi_i(x_i)) = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

□

From Proposition 2.6, we immediately obtain the following Corollary:

Corollary 2.7. (1)

$$\text{in}(\det [\varphi_j(x_i)]) = \prod_{i=1}^{\ell} \text{in}(\varphi_i(x_i)) = \prod_{i=1}^{\ell-1} x_i^{4(\ell-i)}.$$

(2) Moreover, the leading term of $\det [\varphi_j(x_i)]$ is equal to

$$\frac{1}{(2\ell-3)!!} \prod_{i=1}^{\ell-1} x_i^{4(\ell-i)}.$$

(3) In particular, $\det [\varphi_j(x_i)]$ does not vanish.

Next, we will prove $\varphi_j \in D(\mathcal{S}(D_\ell))$ for $1 \leq j \leq \ell$. We denote $\mathcal{S}(D_\ell)$ simply by \mathcal{S}_ℓ from now on. Before the proof, we need the following two lemmas:

Lemma 2.8. Fix $1 \leq j \leq \ell-1$ and $\epsilon \in \{-1, 1\}$. Then

$$(1) \quad \prod_{x_i \in J} (x_i - x_s)(x_i - \epsilon x_t) = \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} \left(\prod K_1 \right) \\ \times \left(\prod K_2 \right)^2 [- (x_s + \epsilon x_t)]^{|K_1|} (\epsilon x_s x_t)^{k_0}.$$

(2)

$$\sum_{\substack{0 \leq n_1 \leq |J_1| \\ 0 \leq n_2 \leq |J_2|}} (-1)^{|J_1|+|J_2|-n_1-n_2} \sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} (\epsilon x_s)^{k+1} \\ = \prod_{x_i \in J_1} (x_i - \epsilon x_s) \prod_{x_i \in J_2} (x_i^2 - x_s^2).$$

Proof. (1) is easy because the left handside is equal to

$$\prod_{x_i \in J} (x_i^2 - (x_s + \epsilon x_t)x_i + \epsilon x_s x_t).$$

(2) The left handside is equal to

$$\sum_{0 \leq n_1 \leq |J_1|} (-\epsilon x_s)^{|J_1|-n_1} \sigma_{n_1}^{J_1} \sum_{0 \leq n_2 \leq |J_2|} (-x_s^2)^{|J_2|-n_2} \tau_{2n_2}^{J_2}$$

which is equal to the right handside. \square

Lemma 2.9.

(1) The polynomial

$$x_s \overline{B}_{k,k_0}(x_s, z) - x_t \overline{B}_{k,k_0}(x_t, z)$$

is divisible by $x_s^2 - x_t^2$,

(2) For $\epsilon \in \{-1, 1\}$, the polynomial

$$(x_s - \epsilon x_t) \epsilon x_s x_t [\overline{B}_{k,k_0}(x_s, z) + \epsilon \overline{B}_{k,k_0}(x_t, z)] \\ - (x_s + \epsilon x_t) (\epsilon x_s x_t)^{k_0} [\epsilon x_t x_s^{k+1} - x_s (\epsilon x_t)^{k+1}]$$

is divisible by $x_s + \epsilon x_t - z$.

Proof. (1) follows from the fact that $-\overline{B}_{k,k_0}(x, z) = \overline{B}_{k,k_0}(-x, z)$ in Proposition 2.1.

(2) follows from the following congruence relation of polynomials modulo the ideal $(x_s + \epsilon x_t - z)$:

$$(x_s - \epsilon x_t) \epsilon x_s x_t [\overline{B}_{k,k_0}(x_s, z) + \epsilon \overline{B}_{k,k_0}(x_t, z)] \\ = (x_s - \epsilon x_t) \epsilon x_s x_t z^{k+2k_0} [B_{k,k_0}(\frac{x_s}{z}) - B_{k,k_0}(\frac{-\epsilon x_t}{z})] \\ \equiv (x_s - \epsilon x_t) \epsilon x_s x_t (x_s + \epsilon x_t)^{k+2k_0} \\ [B_{k,k_0}(\frac{x_s}{x_s + \epsilon x_t}) - B_{k,k_0}(\frac{-\epsilon x_t}{x_s + \epsilon x_t})] \\ = (x_s - \epsilon x_t) \epsilon x_s x_t (x_s + \epsilon x_t)^{k+2k_0} \\ \left(\frac{x_s}{x_s + \epsilon x_t} \right)^k - \left(\frac{\epsilon x_t}{x_s + \epsilon x_t} \right)^k \\ \frac{\left(\frac{x_s}{x_s + \epsilon x_t} \right)^k - \left(\frac{\epsilon x_t}{x_s + \epsilon x_t} \right)^k}{\left(\frac{x_s}{x_s + \epsilon x_t} \right) - \left(\frac{\epsilon x_t}{x_s + \epsilon x_t} \right)} \left(\frac{\epsilon x_t}{x_s + \epsilon x_t} \right)^{k_0} \left(\frac{x_s}{x_s + \epsilon x_t} \right)^{k_0} \\ = (x_s + \epsilon x_t) (\epsilon x_s x_t)^{k_0} [\epsilon x_t x_s^{k+1} - x_s (\epsilon x_t)^{k+1}]. \quad \square$$

Proposition 2.10. Every φ_j lies in $D(\mathcal{S}_\ell)$.

Proof. For $1 \leq j \leq \ell-1$, $1 \leq s < t \leq \ell$, and $\epsilon \in \{-1, 1\}$, by Lemma 2.9 and Lemma 2.8, we have the following congruence relation of polynomials modulo the ideal $(x_s + \epsilon x_t - z)$:

$$(x_s - \epsilon x_t) \epsilon x_s x_t [\varphi_j(x_s + \epsilon x_t - z)] \\ = (x_j - x_{j+1} - z) \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} \left(\prod K_1 \right) \left(\prod K_2 \right)^2 \\ \times (-z)^{|K_1|} \sum_{\substack{0 \leq n_1 \leq |J_1| \\ 0 \leq n_2 \leq |J_2|}} (-1)^{n_1+n_2} \sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \\ \times (x_s - \epsilon x_t) \epsilon x_s x_t [\overline{B}_{k,k_0}(x_s, z) + \epsilon \overline{B}_{k,k_0}(x_t, z)] \\ \equiv (x_j - x_{j+1} - z) (x_s + \epsilon x_t) \\ \times \sum_{K_1, K_2} \left(\prod K_1 \right) \left(\prod K_2 \right)^2 [- (x_s + \epsilon x_t)]^{|K_1|} (\epsilon x_s x_t)^{k_0} \\ \times \sum_{n_1, n_2} (-1)^{n_1+n_2} \sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} [\epsilon x_t x_s^{k+1} - x_s (\epsilon x_t)^{k+1}] \\ = (x_j - x_{j+1} - z) (x_s + \epsilon x_t) \prod_{x_i \in J} (x_i - x_s)(x_i - \epsilon x_t) \\ \times (-1)^{|J_2|} \left[\epsilon x_t \prod_{x_i \in J_1} (x_i - x_s) \prod_{x_i \in J_2} (x_i^2 - x_s^2) \right]$$

$$-x_s \prod_{x_i \in J_1} (x_i - \epsilon x_t) \prod_{x_i \in J_2} (x_i^2 - x_t^2) \quad (\dagger).$$

Case 1. When $x_s \in J$, $(\dagger) = 0$.

Case 2. When $x_s \in J_2$ and $x_t \in J_2$, $(\dagger) = 0$.

Case 3. When $x_s \in J_1$ and $x_t \in J_2$, $(\dagger) = 0$.

Case 4. When $x_s \in J_1$, $x_t \in J_1$ and $\epsilon = 1$, $(\dagger) = 0$.

Case 5. If $x_s \in J_1$, $x_t \in J_1$ and $\epsilon = -1$, then $s = j < t = j + 1$. So (\dagger) is divisible by $x_s + \epsilon x_t - z$.

We also have the following congruence relation of polynomials modulo the ideal $(x_s + \epsilon x_t - z)$:

$$\begin{aligned} & (x_s - \epsilon x_t) \epsilon x_s x_t [\varphi_\ell(x_s + \epsilon x_t - z)] \\ &= \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} \left(\prod K_1 \right) \left(\prod K_2 \right)^2 (-z)^{|K_1|} (-x_\ell) \\ & (x_s - \epsilon x_t) \epsilon x_s x_t [\overline{B}_{-1, k_0}(x_s, z) + \epsilon \overline{B}_{-1, k_0}(x_t, z)] \\ &\equiv (x_s + \epsilon x_t) (-x_\ell) (\epsilon x_t - x_s) \\ & \sum_{K_1, K_2} \left(\prod K_1 \right) \left(\prod K_2 \right)^2 [-(x_s + \epsilon x_t)]^{|K_1|} (\epsilon x_s x_t)^{k_0} \\ &= (x_s^2 - x_t^2) x_\ell \prod_{x_i \in J} (x_i - x_s)(x_i - \epsilon x_t) \quad (\dagger\dagger). \end{aligned}$$

Since $s < t \leq \ell$, we have $x_s \in J = \{x_1, \dots, x_{\ell-1}\}$. Thus $(\dagger\dagger) = 0$. Therefore $\varphi_j(x_s + \epsilon x_t - z)$ is divisible by $x_s + \epsilon x_t - z$ for $1 \leq j \leq \ell, 1 \leq s < t \leq \ell$.

For $1 \leq j \leq \ell$,

$$\varphi_j(x_s^2 - x_t^2) = 2x_s \varphi_j(x_s) - 2x_t \varphi_j(x_t)$$

is divisible either by $x_s \overline{B}_{k, k_0}(x_s, z) - x_t \overline{B}_{k, k_0}(x_t, z)$ or by $x_s \overline{B}_{-1, k_0}(x_s, z) - x_t \overline{B}_{-1, k_0}(x_t, z)$, we have

$$\varphi_j(x_s^2 - x_t^2) \equiv 0 \pmod{(x_s^2 - x_t^2)}$$

by Lemma 2.9 (1). This implies $\varphi_j \in D(\mathcal{S}_\ell)$. \square

Applying Saito's lemma [9] [7, Theorem 4.19], we complete our proof of Theorem 2.4 thanks to Lemma 2.5, Corollary 2.7 (3) and Proposition 2.10. Theorem 2.4 implies that $\det[\varphi_j(x_i)]$ is a nonzero multiple of (Q/z) . By Corollary 2.7 (2) one obtains

Corollary 2.11.

$$\begin{aligned} & \det[\varphi_j(x_i)] \\ &= \frac{1}{(2\ell - 3)!!} \prod_{1 \leq s < t \leq \ell} \prod_{\epsilon \in \{-1, 1\}} (x_s + \epsilon x_t - z)(x_s + \epsilon x_t). \end{aligned}$$

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